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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A least squares solution of sequential array algebra observation equations is derived using spectral decomposition of the normal equation matrix in terms of array algebra. A new direct solution for computation of eigenvectors is derived using the theory of general matrix inverses.

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SEQUENTIAL ARRAY ALGEBRA USING DIRECT SOLUTION OF EIGENVECTORS

PROBLEM OF SEQUENTIAL ARRAY EQUATIONS

The new computationally powerful array algebra technology unifying the sciences of numerical analysis, mathematical statistics and modern signal processing would become more flexible if the problem of sequential array observation equations could be efficiently solved, Rauhala (1974 p 113, 1976 p 79 , 1977, 1978, 1979, 1980a, 1980b), Jancaitis and Magee (1977), Snay (1978). In the illustrative case of three dimensions the sequential observation equations read

$$
\begin{aligned}
E_1 \overset{g_1^T}{X} F_1^T &= L_1 - V_1 \\
E_2 \overset{g_2^T}{X} F_2^T &= L_2 - V_2 \\
&\vdots \\
E_p \overset{g_p^T}{X} F_p^T &= L_p - V_p
\end{aligned}
\qquad (1)
$$

$$d_{j_1 j_2 j_3} \; x_{j_1 j_2 j_3} = (l_d - v_d)_{j_1 j_2 j_3} \; ,$$

where the array multiplications $\underset{m_1 n_1}{E} \; \underset{n_1 n_2 n_3}{\overset{g^T_{n_3 m_3}}{X}} \; \underset{n_2 m_2}{F^T} = \underset{m_1 m_2 m_3}{(L - V)}$ are defined as

$$(l - v)_{r_1 r_2 r_3} = \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} \sum_{j_3=1}^{n_3} (e)_{r_1 j_1} \cdot (f)_{r_2 j_2} \cdot (g)_{r_3 j_3} \; (x)_{j_1 j_2 j_3} \qquad \begin{aligned} r_1 &= 1,2,\cdots m_1 \\ r_2 &= 1,2,\cdots m_2 \\ r_3 &= 1,2,\cdots m_3 . \end{aligned}$$

(2)

The last set of observation equations consists of dot multiplications, i.e., discrete direct observations of parameters $\underset{n_1 n_2 n_3}{X}$ so that in

the conventional monolinear notations where $\underset{N,1}{X}$, $N = n_1 n_2 n_3$, is treated as a long column vector the design matrix would be diagonal.

The above observation equations result in the normal equations

$$\underset{n_1 n_1}{A_1} \underset{n_1 n_2 n_3}{\overset{\underset{n_3 n_3}{C_1}}{X}} \underset{n_2 n_2}{B_1} + A_2 \overset{C_2}{X} B_2 + \cdots A_p \overset{C_p}{X} B_p + D^2 * X = \underset{n_1 n_2 n_3}{W} , \tag{3}$$

where the dot multiplications $d^2_{i_1 i_2 i_3} \; x_{i_1 i_2 i_3}$ are denoted $D^2 * X$.
We now assume that the symmetric square matrices A, B, C are brought to satisfy the following spectral decompositions, for example by using the parameter transformations of Buchanan and Thomas (1968),

$$\begin{aligned}
A_1 &= R^T \alpha_1 R & B_1 &= S^T \beta_1 S & C_1 &= T^T \gamma_1 T \\
A_2 &= R^T \alpha_2 R & B_2 &= S^T \beta_2 S & C_2 &= T^T \gamma_2 T \\
&\vdots & &\vdots & &\vdots \\
A_p &= R^T \alpha_p R & B_p &= S^T \beta_p S & C_p &= T^T \gamma_p T .
\end{aligned} \tag{4}$$

Thus R is the common orthonormal eigenmatrix of all matrices A_i and S, T are its counterparts of matrices B_i , C_i , $i = 1, 2, \cdots p$. The diagonal matrices $\alpha_i , \beta_i , \gamma_i$ contain the eigenvalues of matrices A_i , B_i , C_i.

The present paper is focused on the computational solution of equation (3) under the spectral decomposition of (4). The derivational part of the solution is rather straight-forward, i.e., premultiplications with R , post multiplications with S^T and the "back" multiplications with T^T result in the solution of the diagonal system by

$$R \overset{T^T}{X} S^T = H * R \overset{T^T}{W} S^T \tag{5}$$

$$H = h_{i_1 i_2 i_3} = 1/\{(\alpha_1)_{i_1} (\beta_1)_{i_2} (\gamma_1)_{i_3} + (\alpha_2)_{i_1} (\beta_2)_{i_2} (\gamma_2)_{i_3} + \cdots (\alpha_p)_{i_1} (\beta_p)_{i_2} (\gamma_p)_{i_3} + d_{i_1 i_2 i_3}\} .$$

Now the inverse transformations with R^T, S, T result in the solution familiar from the filtering theory of signal processing

$$\hat{X} = R^T \overset{T}{(H * R \overset{T^T}{W} S^T)} S . \tag{6}$$

In terms of signal processing H can be called "transfer function". In terms of the general theory of linear estimators and matrix inverses, Rauhala (1980b), estimator $\hat{X}$ is unbiased if all $1/h_{j_1 j_2 j_3} \neq 0$. For biased or nearly biased parameters, $h_{j_1 j_2 j_3} \rightarrow \infty$, the bias, variances and the norm of $\hat{X}$ can be minimized through the pseudo-inverse solution simply by putting $h_{j_1 j_2 j_3} = 0$ for $h_{j_1 j_2 j_3} \rightarrow \infty$. All of these solutions of normal equations satisfy the least squares criteria

$$\| \hat{V}_1 \| + \| \hat{V}_2 \| + \cdots \| \hat{V}_p \| + \| \hat{V}_\alpha \| = min. \tag{7}$$

In several applications of array algebra the dimensions n_1, n_2, n_3 of the array $\hat{X}$ can range several hundreds so that the array solution of millions of parameters is split into the problems of solving three small orthonormal eigenmatrices R, S, T. After these matrices are known the array multiplications of equation (6) can be performed along the lines of the computer program presented in (Rauhala, 1980a). The remainder question of this paper handles the computational problem of solving for matrices R, S, T.

DIRECT SOLUTION OF EIGENVECTORS

The computation of eigenvalues λ_i of matrix $\underset{nn}{A}$ and the corresponding eigenvectors is presently dominated by iterative methods putting severe restrictions on the dimensions and conditioning of the matrix. Further the iterative solutions do not guarantee the orthonormality of matrices R_1, R_2 in $\underset{nn}{A} = \underset{nn}{R_1}^T \underset{nn}{\lambda} \underset{nn}{R_2}$.

In the new direct approach of finding R_1, R_2 we split the eigenvalue problem in two separate parts, i.e., we assume that the eigenvalue λ_i is known or computed a priori. We are seeking direct solutions for the corresponding eigenvectors $\underset{n,1}{x_i}$, $\underset{1,n}{y_i^T}$ as the non-homogeneous solution of the consistent systems

$$A_i \quad x_i = 0 \tag{8a}$$

$$y_i^T \quad A_i = 0 , \tag{8b}$$

where

$$A_i = A - \lambda_i I . \tag{9}$$

The solutions are found using the general theory of matrix inverses, Rauhala (1980b), by

$$\hat{x}_i = (I - A_i^g A_i)\, u_1 \tag{10a}$$

$$y_i^T = u_2^T (I - A_i A_i^g) . \tag{10b}$$

Vectors u_1, u_2^T are arbitrary and the g-inverse A_i^g needs to satisfy the condition $A_i A_i^g A_i = A_i$ in order to have (10a), (10b) as the solutions of (8a), (8b).

Because by the definition $\det|A_i| = \det|A - \lambda_i I| \equiv 0$ the maximum rank of matrix $\underset{nn}{A_i}$ is $r = n-1$. We perform the rank factorization of A_i as

$$\underset{nn}{A_i} = \begin{bmatrix} \underset{rr}{A_0} & \underset{r,n-r}{A_1} \\ \underset{n-r,r}{A_2} & \underset{n-r,n-r}{A_3} \end{bmatrix}, \tag{11}$$

where the submatrix A_3 has to satisfy the condition $A_3 = A_2 A_0^{-1} A_1$.

This condition can be derived by eliminating the "independent" parameters $\underset{r,1}{Z_1}$ from the system

$$A_0 Z_1 + A_1 Z_2 = W_1 \tag{12a}$$

$$A_2 Z_1 + A_3 Z_2 = W_2 \tag{12b}$$

by

$$Z_1 = A_0^{-1}(W_1 - A_1 Z_2) \tag{13}$$

Substitution into the linearly dependent part of (12b), yields

$$(A_3 - A_2 A_0^{-1} A_1) Z_2 = W_2 - A_2 A_0^{-1} W_1$$

$$\therefore \quad A_3 = A_2 A_0^{-1} A_1 . \tag{14}$$

The computational rule of finding A_i^g of (11) is simply

$$\underset{nn}{A_i^g} = \begin{bmatrix} \underset{r,r}{A_0^{-1}} & \underset{r,n-r}{0} \\ \underset{n-r,r}{0} & \underset{n-r,n-r}{0} \end{bmatrix}, \tag{15}$$

because it satisfies the condition $A_i A_i^g A_i = A_i$ yielding

$$A_i^g A_i = \begin{bmatrix} I & A_0^{-1}A_1 \\ 0 & 0 \end{bmatrix} \rightarrow I - A_i^g A_i = \begin{bmatrix} \underset{r,r}{0} & -A_0^{-1}A_1 \\ 0 & \underset{n-r,n-r}{I} \end{bmatrix} \tag{16a}$$

$$A_i A_i^g = \begin{bmatrix} I & 0 \\ A_2 A_0^{-1} & 0 \end{bmatrix} \rightarrow I - A_i A_i^g = \begin{bmatrix} \underset{r,r}{0} & 0 \\ -A_2 A_0^{-1} & \underset{n-r,n-r}{I} \end{bmatrix}. \tag{16b}$$

The unnormalized eigenvector solutions become now from (10a), (10b)

$$\underset{n,1}{x_i} = \begin{bmatrix} -A_0^{-1} \underset{r,1}{\Sigma_1} \\ 1 \\ \vdots \\ 1 \end{bmatrix} \qquad (\Sigma_1)_j = \sum_{k=1}^{n-r} (A_1)_{jk} \quad j = 1, 2, \cdots r. \tag{17a}$$

$$\underset{1,n}{y_i^T} = \begin{bmatrix} -\underset{1,r}{\Sigma_2} A_0^{-1}, & 1, & 1, \cdots 1 \end{bmatrix} \qquad (\Sigma_2)_j = \sum_{k=1}^{n-r} (A_2)_{kj}, \tag{17b}$$

where the $n-r$ last terms of $\underset{n,1}{u_1}, \underset{1,n}{u_2^T}$ are chosen to be ones. The normalized eigenvectors become

$$\underset{n,1}{\hat{x}_i} = \left(n-r+\sum_{j=1}^{r}(k_x)_j^2\right)^{-1/2} \begin{bmatrix} \underset{r,1}{k_x} \\ 1 \\ \vdots \\ 1 \end{bmatrix} \qquad \underset{r,1}{k_x} = -\underset{r,r}{A_0^{-1}} \underset{r,1}{\Sigma_1} \tag{18a}$$

$$\underset{1,n}{y_i^T} = \left(n-r+\sum_{j=1}^{r}(k_y)_j^2\right)^{-1/2} \begin{bmatrix} k_y^T, 1, 1, \cdots 1 \end{bmatrix}, \quad \underset{1,r}{k_y^T} = -\underset{1,r}{\Sigma_2} \underset{r,r}{A_0^{-1}}. \tag{18b}$$

By repeating these solutions for all eigenvalues $\lambda_i, i=1,2,\dots n$ the sought orthonormalized matrices R_1, R_2 of $A = R_1^T \lambda R_2$ become

$$\underset{n,n}{R_1^T} = \left[\hat{x}_1, \hat{x}_2, \cdots \hat{x}_n\right], \underset{n,n}{R_2} = \begin{bmatrix} \hat{y}_1^T \\ \hat{y}_2^T \\ \vdots \\ \hat{y}_n^T \end{bmatrix} . \tag{19}$$

For a symmetric matrix $R_1 = R_2 = R$.

OPERATION COUNT OF THE DIRECT SOLUTION

Computation of R for a symmetric non-sparse matrix requires in the order of $r^3 \simeq n^3$ operations (scalar multiplications and additions) for each $\underset{r,1}{(k_x)_i} = \underset{r,r}{A_0^{-1}} \underset{r,1}{\Sigma_1}$, or totally n^4 operations. In our array algebra solution this operation count is by no means prohibitive as we are solving for $N = n_1 n_2 n_3 \simeq n^3$ parameters. In fact, the three-dimensional array multiplications using the general non-sparse matrices R, S, T in equation (6) require the same magnitude of n^4 operations, Rauhala (1976, 1979, 1980a), Blaha (1977).

If the spectral decomposition of the $n-1, n-1$ leading partition of A were known as $\tilde{R}^T \tilde{\lambda} \tilde{R}$ and the same partition could be used for all $(k_x)_i$ then we could perform the one-time multiplication $\underset{n-1,1}{\tilde{\Sigma}_1} = \underset{n-1,n-1}{\tilde{R}} \underset{n-1,1}{\Sigma_1}$ and each $(k_x)_i = \tilde{R}^T(\tilde{\lambda} - \lambda_i I)^{-1}\tilde{\Sigma}_1$ would require n^2 operations or totally $\underset{n,n}{R}$ would require n^3 operations. This is the same magnitude of operations required for a two-dimensional array multiplication of the type of equation (6).

In several practical applications matrix $\underset{n,n}{A}$ is banded and only a

few last terms of A_1 are non-zeroes so that each $(k_x)_i$ requires $b^2 n$ operations or totally $b^2 n^2$ operations for $\underset{nn}{R}$, where b is the bandwidth (usually $b \leq 5$ for symmetric matrices). In some practical experiments the author performed the double precision orthonormalization of a 300 X 300 tridiagonal matrix in a CPU time of a few seconds using a minicomputer.

APPLICATIONS

The above array solutions were used for simulations of non-separable filters of finite element solution of regularly gridded data. Using these filters or impulse responses a rigorous least squares solution of 601 X 1201 > 720 000 nodes was convolved in a CPU time of less than one minute and using less than 30 K bytes of the minicomputer core space.

For the non-stationary case of irregular gridded data the above derived array solution (6) removes some restrictions of the one-batch array equations. For example in digital terrain, geoid, gravity anomaly etc. modeling using the method of array algebra finite elements the observed nodes are allowed to have completely arbitrary locations and a priori weights. Simultaneously the operators R, S, T can be brought to exhibit a structure of generalized fast transforms, (Rauhala (1980a), so that R, S, T are never explicitely computed (requiring no core space) and multicplication $\underset{nn}{R}\,\underset{n,1}{W}$ requires less than $\log_2 n \cdot n$ operations, i.e., the total solution of $N = n_1 n_2 n_3 = n^3$ parameters requires the magnitude of N operations.

The above very fast array solution can exhibit such general

properties which seem to be at odds with the restrictive nature of each sequential array batch: For example, the math model can contain equality constraints, discontinuities and break-lines and single point constraints (minimum, maximum, saddle etc. points). Furthermore, the math model allows automatic bridging of "smooth areas" (sparse data sampling) or a priori identified "blunder areas" (sampled data with zero a priori weights). Those features allow introduction of batches of fill-in samples replacing large areas of blunderous observations, batches of overlapping data samples, etc. Thus the math model can be used for modeling even "pathologically" difficult and ill-behaving empirical functions with proper computational efficiency both in the stages of forming the data base and in the retrieval and usage of the stored data base.

The above and many other applications of the sequential array algebra warrant detailed investigations. For example some carefully designed net adjustment problems of large dimensions are within the capabilities of the above array solution.

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